Lecture 12

4. $Q(x)$ has repeated irreducible quadratic factors.
For every factor of the form, $(a_ix^2+b_ix+c_i)^k$, $k>1$, we include the terms

$$\frac{A_1x+B_1}{a_ix^2+b_ix+c_i}+\cdots+\frac{A_kx+B_k}{(a_ix^2+b_ix+c_i)^k}$$

For example, the decomposition of:

$$\frac{6x^4-2x+1}{x^8+3x^6+3x^4+x^2}=\frac{6x^4-2x+1}{x^2(x^2+1)^3}=\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{B_1x+C_1}{x^2+1}+\frac{B_2x+C_2}{(x^2+1)^2}+\frac{B_3x+C_3}{(x^2+1)^3}$$

Ex: Compute $\displaystyle\int\frac{dx}{x(x^2+4)^2}$

$$\frac{1}{x(x^2+4)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}$$

$$\Rightarrow 1=A(x^2+4)^2+(Bx+C)x(x^2+4)+(Dx+E)x$$

$$=Ax^4+8Ax^2+16A+Bx^4+4Bx^2+Cx^3+4Cx+Dx^2+Ex$$

$$=(A+B)x^4+(C)x^3+(8A+4B+D)x^2+(4C+E)x+(16A)$$

$$\begin{cases} A+B=0 & ① \\ C=0 & ② \\ 8A+4B+D=0 & ③ \\ 4C+E=0 & ④ \\ 16A=1 & ⑤ \end{cases}$$

② $C=0 \overset{④}{\Rightarrow} E=0$ | ⑤ $\Rightarrow A=\frac{1}{16}$

① $\Rightarrow B=\frac{-1}{16}$

③ $\Rightarrow D=-8A-4B=\frac{-1}{2}+\frac{1}{4}=\frac{-1}{4}$

$$\int\frac{dx}{x(x^2+4)^2}=\int\left(\frac{1/16}{x}+\frac{\frac{-1}{16}x}{x^2+4}+\frac{\frac{-1}{4}x}{(x^2+4)^2}\right)dx=\boxed{\frac{1}{16}\ln|x|-\frac{1}{32}\ln|x^2+4|-\frac{1}{8}\frac{1}{x^2+4}+C}$$

$u=x^2+4 \quad \downarrow \quad \frac{-1}{32}\int\frac{1}{u}du$

$u=x^2+4 \quad \downarrow \quad \frac{1}{8}\int\frac{1}{u^2}du$

If the numerator is a polynomial of higher degree than the denominator, then we must do long division first to write

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

so $\deg(R) < \deg(Q)$

Ex: Compute $\displaystyle\int \frac{x^3-4x-10}{x^2-x-6}\,dx$

$$\begin{array}{r}
x+1 \\
(x^2-x-6)\overline{\big)\,x^3+0x^2-4x-10} \\
-x^3+x^2+6x \\
\hline
x^2+2x-10 \\
-x^2+x+6 \\
\hline
3x-4 \;\; R
\end{array}$$

$$\Rightarrow \frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{3x-4}{x^2-x-6} = x+1+\frac{3x-4}{(x-3)(x+2)}$$

$$\frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3}+\frac{B}{x+2}$$

$$\Rightarrow 3x-4 = A(x+2)+B(x-3) = (A+B)x+(2A-3B)$$

$$\begin{cases} A+B=3 & ① \\ 2A-3B=-4 & ② \end{cases} \qquad 3①+②: 5A=5 \Rightarrow A=1 \Rightarrow B=2$$

$$\int \frac{x^3-4x-10}{x^2-x-6}\,dx = \int \left(x+1+\frac{1}{x-3}+\frac{2}{x+2}\right)dx = \frac{1}{2}x^2+x+\ln|x-3|+2\ln|x+2|+C$$

$$= \boxed{\frac{1}{2}x^2+x+\ln\left|(x-3)(x+2)^2\right|+C}$$

Sometimes it may be necessary to make a substitution before attempting partial fractions.

Ex: Compute $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$

Let $u=e^x$, then $du=e^x dx$ and

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx = \int \frac{u}{u^2+3u+2}\,du = \int \frac{u}{(u+2)(u+1)}\,du$$

$$\frac{u}{(u+2)(u+1)} = \frac{A}{u+2}+\frac{B}{u+1} \Rightarrow u = A(u+1)+B(u+2) = (A+B)u+(A+2B)$$

$$\begin{cases} A+B=1 \\ A+2B=0 \Rightarrow A=-2B \Rightarrow -B=1 \Rightarrow B=-1 \Rightarrow A=2 \end{cases}$$

$$\int \frac{u}{(u+2)(u+1)}\,du = \int \left(\frac{2}{u+2}+\frac{-1}{u+1}\right)du = 2\ln|u+2| - \ln|u+1| + C$$

$$= \ln\left|\frac{(u+2)^2}{u+1}\right| + C \overset{u=e^x}{=\!=} \boxed{\ln\left|\frac{(e^x+2)^2}{e^x+1}\right| + C}$$

Ex: Evaluate $\int \frac{dx}{x^2+x\sqrt{x}}$

Let $u=\sqrt{x}$. Then $du=\frac{1}{2}\frac{1}{\sqrt{x}}dx \Rightarrow 2du=\frac{1}{\sqrt{x}}dx$

$$\int \frac{dx}{x^2+x\sqrt{x}} = \int \left(\frac{1}{(\sqrt{x})^3+(\sqrt{x})^2}\right)\frac{dx}{\sqrt{x}} = \int \frac{2}{u^3+u^2}du = \int \frac{2}{u^2(u+1)}du$$

$$\frac{2}{u^2(u+1)} = \frac{A}{u}+\frac{B}{u^2}+\frac{C}{u+1}$$

$$\Rightarrow 2 = Au(u+1)+B(u+1)+Cu^2 = (A+C)u^2+(A+B)u+B$$

$$\Rightarrow \begin{cases} A \quad\quad +C = 0 & ① \\ A+B \quad\quad = 0 & ② \\ \quad\quad B \quad\quad = 2 & ③ \end{cases}$$

③: $B=2 \overset{②}{\Rightarrow} A=-2 \overset{①}{\Rightarrow} C=2$

$$\int \frac{du}{x^2+x\sqrt{x}} = \int \frac{2}{u^2(u+1)}du = \int \left(\frac{-2}{u}+\frac{2}{u^2}+\frac{2}{u+1}\right)du$$

$$= -2\ln|u| - \frac{2}{u} + 2\ln|u+1| + C$$

$$= \ln\left[\left(\frac{u+1}{u}\right)^2\right] - \frac{2}{u} + C = \boxed{\ln\left[\left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)^2\right] - \frac{2}{\sqrt{x}} + C}$$